L-functions of degree 4 and weight 1

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joint work with
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Computational Aspects of L-functions
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Goal: To make a catalogue of all L-functions.

In particular:

- degree 4, weight 1, rational integer coefficients
  \((arithmetic\ \text{normalisation})\)

Restrictive, but not too restrictive:

- products of elliptic curves over \(\mathbb{Q}\)
- genus 2 curves over \(\mathbb{Q}\)
- elliptic curves over quadratic fields \(\mathbb{Q}(\sqrt{d})\)
- Siegel modular forms
- Hilbert modular forms
- Bianchi modular forms
- Abelian surfaces over \(\mathbb{Q}\)
L-functions:

degree 4, weight 1, rational integer coefficients

- Dirichlet series
  \[ L(s) = \sum_{n=1}^{\infty} \frac{A_n/\sqrt{n}}{n^s} \quad A_n \in \mathbb{Z} \]

- Functional equation
  \[ \Lambda(s) := N^{s/2} \Gamma_C(s + \frac{1}{2})^2 L(s) = \pm \Lambda(1 - s) \]

- Euler product
  \[ L(s) = \prod_p F_p(p^{-s})^{-1}, \text{ where} \]
  \[ F_p(z) = G_p(z/\sqrt{p}) \text{ with } G_p(z) \in \mathbb{Z}[z] \]

Furthermore
- \( F_p(0) = 1 \)
- If \( p \nmid N \) then \( F_p(z) \) has degree 4.
  Also: all roots of \( F_p(z) \) lie on \( |z| = 1 \).
- If \( p|N \) then \( F_p(z) \) has degree \( \leq 3 \).
  Also: each root of \( F_p(z) \) lies on \( |z| = p^{m/2} \)
  for some \( m \in \{0, 1, 2, 3\} \).
There are only finitely many choices for $F_p$ for each $p$.

**Table:** Number of possible local factors

<table>
<thead>
<tr>
<th>prime, $p$</th>
<th>good</th>
<th>bad</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>35</td>
<td>26</td>
</tr>
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<td>3</td>
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<td>38</td>
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<td>7</td>
<td>207</td>
<td>44</td>
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</table>

**The Method:**
For a given $N$ and $\varepsilon = \pm 1$, everything about the L-function is known except the coefficients.

→ find coefficients such that the functional equation is true

OR

prove that the L-function cannot exist.

→ This involves searching a tree.
Searching a tree:
Preliminary results

For degree 4, weight 1 L-functions with rational integer coefficients:

\[(N, \varepsilon) \mapsto \text{specific functional equation} + \text{the approx. F.E.} \mapsto \text{equation in ‘L-function’ coefficients}\]

Search:

- \(N: \leq 680\)
- \(\varepsilon: \pm 1\)

results:

Table of L-functions
Some questions

1. What genus 2 curve or abelian surface has conductor 550? 
   *(We can tell you the first 200 coefficients.)*

2. Why are the first several L-functions non-primitive?

3. Will the primitive L-functions eventually be ‘most’ of them?